

SUBGROUPS OF FINITELY PRESENTED SOLVABLE LINEAR GROUPS

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ABSTRACT. Let G be a finitely generated solvable linear group. It is shown that there exists a finitely presented solvable linear group H with G embedded in H .

1.1. The complexity of the subgroup structure of finitely presented solvable groups has only recently begun to emerge (e.g. see the papers of Gilbert Baumslag [2]–[6]). Among the results is the following theorem of Baumslag [2]: *Every finitely generated metabelian group can be embedded in a finitely presented metabelian group.* This indicates that the subgroup structure of finitely presented solvable groups is surprisingly complex. In fact, in [3] Baumslag has demonstrated the existence of a finitely presented solvable group which is not residually finite.

Despite this knowledge many open problems remain, e.g.: *Is a finitely presented solvable group hopfian?* In this regard it is worth noting that Philip Hall's old question (see [11]) as to whether a finitely presented solvable group necessarily satisfies the maximal condition for normal subgroups is still open—V. N. Remeslennikov's assertion in [14] is unfortunately incorrect because his group does satisfy the maximal condition for normal subgroups.

The object of this paper is to throw some additional light on the nature of finitely presented solvable groups by proving the following:

THEOREM 1. *Let R be a commutative associative ring with 1 and n a positive integer. Further, let G be a finitely generated multiplicative group of $n \times n$ triangular matrices over R ⁽¹⁾. Then there exists a commutative associative ring S with 1 and a finitely presented multiplicative group H of $n \times n$ triangular matrices over S with G embeddable in H . Moreover if R is also a domain, then S can be taken to be the field of fractions of R .*

(The analogue of this theorem for associative algebras has recently been obtained by Gilbert Baumslag [7].)

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(¹) By definition a matrix is triangular if all its elements above the main diagonal are zero.

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Remeslennikov [15] has pointed out that if G is any finitely generated metabelian group there exists a commutative associative ring R with 1 such that G is embedded in the multiplicative group of 2×2 triangular matrices over R . Hence for n equal to 2, Theorem 1 reduces to Baumslag's Theorem [2].

1.2. Theorem 1 admits a slight, but pleasing, generalization for solvable linear groups:

THEOREM 2. *Every finitely generated solvable linear group can be embedded in a finitely presented solvable linear group.*

Here a linear group is simply a multiplicative group of nonsingular matrices with coefficients in some commutative field.

There are some other consequences of Theorem 1 that are worth observing. First of all we note that every countable free nilpotent group of any given class c has a faithful representation as a linear group (e.g. see B. A. F. Wehrfritz [17, p. 34]). It is not difficult to embed this linear representation into a finitely generated solvable linear group and thereby prove

COROLLARY 1. *A free nilpotent group of any given class c and countable rank can be embedded in a finitely presented solvable linear group.*

Similarly using the appropriate matrix representations due to D. I. Èidel'kind [8] in the first case and C. K. Gupta [9] in the second, one can also prove the further corollaries.

COROLLARY 2. *A finitely generated free group in the product variety $\mathfrak{N}_c \mathfrak{A}$ can be embedded in a finitely presented linear group in the same variety (here \mathfrak{N}_c is the variety of all nilpotent groups with class less than or equal to c and \mathfrak{A} is the variety of all abelian groups).*

COROLLARY 3. *The free center-by-metabelian group of rank 3 can be embedded in a finitely presented solvable linear group.*

It should be noted that the result of D. I. Èidel'kind [8] was also proved by N. S. Romanovskii [16] and C. K. Gupta and N. D. Gupta [10].

1.3. The arrangement of this paper is as follows. First we shall take care of the notation and all the other preliminaries in §2. The proof of Theorem 1 begins in §3 and is concluded in §7. The proofs of Theorem 2 and Corollary 1 are contained in §§8 and 9 respectively. Finally, in §10 we shall explain why Remeslennikov's assertion in [14] is incorrect.

2. In this section we record some of the notation and notions we shall need in the sequel.

As usual if a and b are elements of a group G and H is a subgroup of G then we define

$$[a, b] = a^{-1}b^{-1}ab, \quad a^b = b^{-1}ab \quad \text{and} \quad H^b = b^{-1}Hb.$$

If t is an element of G and $f(x) = n_0 + n_1x + \cdots + n_dx^d$ is an element of the polynomial ring $Z[x]$ then we shall write

$$a^{f(t)} = a^{n_0}(a^{n_1})^t \cdots (a^{n_d})^{t^d}.$$

We shall need two lemmas of Baumslag [2], which we record here as Lemma 1 and Lemma 2.

LEMMA 1. *Let R be a finitely generated commutative associative ring with 1 and let s be a unit of R with infinite order. Then there exists a nonzero divisor f in R of the form*

$$f = 1 + c_1s + \cdots + c_{d-1}s^{d-1} + s^d$$

where d, c_1, \dots, c_{d-1} are integers and d is positive.

LEMMA 2. *Let a, b, t, u be elements of a group and let d be a positive integer. Suppose that $[t, u] = 1$ and that $[a, b^{t^n}] = 1$ whenever $-d \leq n \leq d$. In addition suppose that $a^u = a^{f(t)}, b^u = b^{f(t)}$ where $f(x) = 1 + c_1x + \cdots + c_{d-1}x^{d-1} + x^d$ is an element of the polynomial ring $Z[x]$. Then $[a, b^{t^i u^j}] = 1$ for all integers i, j .*

Let R be an associative ring with 1 and n a positive integer. $\text{GL}(n, R)$ denotes the group of units of the ring of $n \times n$ matrices over R with identity 1_n . The set of triangular matrices in $\text{GL}(n, R)$ forms the subgroup of $\text{GL}(n, R)$ which is denoted by $\text{Tr}(n, R)$. As usual if A is an element of $\text{GL}(n, R)$ we will write $A = (a_{i,j})$ where $a_{i,j}$ is the entry appearing in the i th row and j th column of the matrix A .

Throughout the remainder of this paper a ring will be taken to mean a commutative associative ring with a 1. The facts about these rings which are used in §4 may be found in Atiyah and Macdonald [1].

3. Our ultimate objective is to show that if R is a ring, n a positive integer and G a finitely generated subgroup of $\text{Tr}(n, R)$ then there exist a ring S and a finitely presented subgroup H of $\text{Tr}(n, S)$ with G embeddable in H . Moreover if R is also a domain, then S can be taken to be the field of fractions of R .

The proof of these assertions is divided into four sections. The first section deals with constructing a suitable ring to play the role of S . Secondly, we examine a special subgroup H of $\text{Tr}(n, S)$ which contains an isomorphic copy of G . The final step of the proof is devoted to showing that H is a finitely presented group.

4. Let R be a ring, n a positive integer and G a finitely generated subgroup of $\text{Tr}(n, R)$. Suppose G is generated by the matrices g_1, \dots, g_m .

We can assume that R is finitely generated. Thus, by the Lasker-Noether theorem, there exists a primary decomposition of the zero ideal, say $q_1 \cap \dots \cap q_p = (0)$. Let R_i be the quotient ring R/q_i ($1 \leq i \leq p$). Then R is embedded in the direct sum R' of the rings R_i . Since the zero divisors of R_i are nilpotent elements, by localizing the nonzero divisors of R_i if necessary, we can assume every element of R_i is a unit or a nilpotent element ($1 \leq i \leq p$). It follows that for our purposes we can assume the additive group of R' is generated by units of R' . Therefore we can choose a finite set of units s_1, \dots, s_q of R' such that the elements of R' appearing below the main diagonal of the generating matrices g_1, \dots, g_m of G are all contained in the additive subgroup of R' generated by s_1, \dots, s_q . In addition let s_{q+1}, \dots, s_r be the units of R' which appear on the main diagonal of the generating matrices g_1, \dots, g_m of G .

Now let R'' be the subring with 1 of R' generated by the elements $s_1, \dots, s_q, s_{q+1}, \dots, s_r$ and their inverses. By relabeling if necessary, we can assume that s_1, \dots, s_q are of infinite order and that s_{q+1}, \dots, s_r are respectively of finite order e_{q+1}, \dots, e_r . According to Lemma 1 for each s_i of infinite order there exists a nonzero divisor f_i in R'' of the form

$$(1) \quad f_i = 1 + c_{i,1}s_i + \dots + c_{i,d_i-1}s_i^{d_i-1} + s_i^{d_i}$$

where $d_i, c_{i,1}, \dots, c_{i,d_i-1}$ are integers and d_i is positive ($1 \leq i \leq q$). Therefore, by localizing R'' if necessary, we can assume f_i is a unit in R'' ($1 \leq i \leq q$). Let S be the subring with 1 of R'' generated by the elements $s_1, \dots, s_q, s_{q+1}, \dots, s_r, f_1, \dots, f_q$ and their inverses.

In the event that R is a domain, let R'' be its field of fractions and $s_1, \dots, s_q, s_{q+1}, \dots, s_r$ be the collection of all nonzero entries appearing in the generating matrices g_1, \dots, g_m of G . Again we may assume that s_1, \dots, s_q are of infinite order and that s_{q+1}, \dots, s_r are respectively of finite order e_{q+1}, \dots, e_r . Put $f_i = 1 + s_i$ ($1 \leq i \leq q$) and then let S be the subring with 1 of R'' generated by the elements $s_1, \dots, s_q, s_{q+1}, \dots, s_r, f_1, \dots, f_q$ and their inverses.

5. We need a special subgroup of $\text{Tr}(n, S)$, viz,

$$G(n, S) = \{(a_{i,j}) \in \text{Tr}(n, S) | a_{i,i} \in S^* (1 \leq i \leq n)\},$$

where S^* is the multiplicative subgroup of S generated by the elements $s_1, \dots, s_q, s_{q+1}, \dots, s_r, f_1, \dots, f_q$. By the construction of S we see that G is embedded in $G(n, S)$.

Notice that $G(1, S)$ is isomorphic to S^* . Let $n \geq 2$. If A is an element of $G(n, S)$ then we can write

$$A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & & & \\ \vdots & & A' & \\ a_n & & & \end{pmatrix},$$

where $A' \in G(n-1, S)$ is simply the lower-right $(n-1) \times (n-1)$ submatrix of A . By straightforward multiplication of the lower triangular matrices we see that the mapping θ given by $\theta: A \mapsto A'$ defines a homomorphism of $G(n, S)$ onto $G(n-1, S)$ with kernel

$$K(n, S) = \left\{ A \in G(n, S) \mid A = \begin{pmatrix} a_1 & 0 & \cdots & 0 \\ a_2 & & & \\ \vdots & & 1_{n-1} & \\ a_n & & & \end{pmatrix} \right\}.$$

Thus for $n \geq 2$ we have a short exact sequence of groups

$$1 \rightarrow K(n, S) \rightarrow G(n, S) \rightarrow G(n-1, S) \rightarrow 1$$

with $G(1, S)$ isomorphic to S^* .

6. We shall need, in the sequel, an analysis of $K(n, S)$. Let $n \geq 2$. Then for each i ($2 \leq i \leq n$) and $a \in S$ let $A_i(a)$ be the $n \times n$ lower triangular matrix with entry a in the i th row and first column, 1 everywhere on the main diagonal, zero elsewhere. For each $b \in S^*$ let $D(b)$ denote the $n \times n$ diagonal matrix with entry b in the first position on the main diagonal and the entry 1 thereafter. Obviously these matrices are contained in $K(n, S)$. We list some of the easily verified properties of these matrices:

$$\begin{aligned} (2) \quad & D(b_1)^{n_1} \cdots D(b_r)^{n_r} = D(b_1^{n_1} \cdots b_r^{n_r}) \quad (n_1, \dots, n_r \in \mathbb{Z}), \\ & [D(b), D(b')] = 1_n, \\ & A_i(a_1)^{n_1} \cdots A_i(a_r)^{n_r} \\ & \quad = A_i(n_1 a_1 + \cdots + n_r a_r) \quad (n_1, \dots, n_r \in \mathbb{Z}, 2 \leq i \leq n), \\ & [A_i(a), A_{i'}(a')] = 1_n \quad (2 \leq i, i' \leq n), \\ & A_i(a)^{D(b)} = A_i(ab) \quad (2 \leq i \leq n), \end{aligned}$$

and

$$D(b)A_2(a_2) \cdots A_n(a_n) = \begin{pmatrix} b & 0 & \cdots & 0 \\ a_2 & & & \\ \vdots & & 1_{n-1} & \\ a_n & & & \end{pmatrix}.$$

If we put $A_i = A_i(1)$ for each i ($2 \leq i \leq n$), it follows that we have proved that

$$(3) \quad K(n, S) = \text{gp}(A_2, \dots, A_n, D(s_i), D(f_j) | 1 \leq i \leq r, 1 \leq j \leq q)$$

is a finitely generated metabelian group.

7. We are now ready to prove that $G(n, S)$ is finitely presented. Recall from §5 that $G(1, S)$ is isomorphic to S^* and for each $n \geq 2$ we have a short exact sequence of groups

$$1 \rightarrow K(n, S) \rightarrow G(n, S) \rightarrow G(n-1, S) \rightarrow 1.$$

It is clear that $G(1, S)$ is finitely presented. Inductively, for $n \geq 2$, we shall assume that $G(n-1, S)$ is finitely presented. Then, in view of the displayed short exact sequence of groups above, to show $G(n, S)$ is finitely presented it will be sufficient to show that $K(n, S)$ is finitely presented for each $n \geq 2$.

Let $n \geq 2$ be fixed. To show $K(n, S)$ is finitely presented we shall construct a finitely presented metabelian group M which has $K(n, S)$ as a homomorphic image. Since the class of finitely presented metabelian groups is an image closed class (Hall [11]), this will insure that $K(n, S)$ is finitely presented.

We shall present M on the generators

$$(4) \quad X = \{a_2, \dots, a_n, t_1, \dots, t_q, t_{q+1}, \dots, t_r, u_1, \dots, u_q\}$$

where the integers n, q, r are the integers occurring in the description of $K(n, S)$ (see (3)).

The defining relations of M are of four kinds. First we have the power relations

$$(5) \quad t_i^{e_i} = 1 \quad (q+1 \leq i \leq r)$$

where here the positive integer e_i is the order of the element s_i ($q+1 \leq i \leq r$). Next we have the commutativity relations

$$(6) \quad \begin{aligned} [t_i, t_j] &= 1 & (1 \leq i, j \leq r), \\ [t_i, u_j] &= 1 & (1 \leq i \leq r, 1 \leq j \leq q), \\ [u_i, u_j] &= 1 & (1 \leq i, j \leq q). \end{aligned}$$

Thirdly we have the commutativity relations for the conjugates of the generators a_i :

$$(7) \quad \begin{aligned} [a_i, a_j^w] &= 1 \quad \text{where } 2 \leq i, j \leq n, \text{ and} \\ w &\in \{t_1^{n_1} \cdots t_q^{n_q} t_{q+1}^{n_{q+1}} \cdots t_r^{n_r} \mid -d_i \leq n_i \leq d_i \ (1 \leq i \leq q), \\ &\quad 0 \leq n_i < e_i \ (q+1 \leq i \leq r)\}. \end{aligned}$$

Here the positive integer d_i occurring in (7) is simply the "degree" of the element f_i (see (1)) ($1 \leq i \leq q$). Finally we have the defining relations giving the action of the elements u_j on the elements a_i :

$$(8) \quad a_i^{u_j} = a_i^{f_j(t_j)} \quad (2 \leq i \leq n, 1 \leq j \leq q),$$

where $f_j(x) = 1 + c_{j,1}x + \cdots + c_{j,d_j-1}x^{d_j-1} + x^{d_j}$ is an element of the polynomial ring $Z[x]$ (again see (1)).

We emphasize that M is the group generated by the set X given by (4) subject to the defining relators (5), (6), (7) and (8). M is patently finitely presented. Moreover, upon appealing to Lemma 2, M is also metabelian.

Now define a map θ from M to $K(n, S)$ by the rule

$$\theta: \begin{cases} a_i \mapsto A_i & (2 \leq i \leq n), \\ t_i \mapsto D(s_i) & (1 \leq i \leq r), \\ u_i \mapsto D(f_i) & (1 \leq i \leq q). \end{cases}$$

It follows, upon examining (1), the relations satisfied in $K(n, S)$ (see (2)) and the defining relations (5)–(8) of M , that θ extends to a homomorphism from M onto $K(n, S)$. Hence $K(n, S)$ is finitely presented as desired.

Thus we have proved Theorem 1.

It is worth noting that the group $G(n, S)$ of this theorem actually has a normal series of subgroups

$$1 = H_0 \leq H_1 \leq \cdots \leq H_n = G(n, S)$$

in which the successive factor groups H_{i+1}/H_i ($i = 0, \dots, n-1$), are all finitely presented and metabelian. In particular $G(n, S)$ satisfies the maximal condition for normal subgroups.

8. It is now easy to prove

THEOREM 2. *Every finitely generated solvable linear group can be embedded in a finitely presented solvable linear group.*

PROOF. Let G be a finitely generated solvable subgroup of $\text{GL}(n, k)$. We can suppose that k is algebraically closed. Hence there exists a subgroup H of G

with finite index and an element x of $\text{GL}(n, k)$ such that H^x is a subgroup of $\text{Tr}(n, k)$ (A. I. Mal'cev [13]). Replacing G by G^x if necessary, we can assume that H is a subgroup of $\text{Tr}(n, k)$. Since G is finitely generated and H has finite index in G , H is also finitely generated and we can even assume that H is a normal subgroup of G . Thus H is a subgroup of a finitely presented subgroup K of $\text{Tr}(n, k)$. Since G/H is finite, G can be embedded in the standard (restricted) wreath product $H \sim G/H$ of H by G/H (see Krasner and Kaloujnine [12]). Let $W = K \sim G/H$ be the standard wreath product of K by G/H and notice that $H \sim G/H$ is a subgroup of W . Therefore G has been embedded in W .

We observe that since K is finitely presented and G/H is finite it follows that W is finitely presented. Next, since W is the wreath product of solvable groups, it too is solvable. Finally, K is a subgroup of $\text{GL}(n, k)$ and G/H is finite so that W is isomorphic to a linear group over k (see B. A. F. Wehrfritz [17, p. 151]).

9. In this section we shall prove Corollary 1 by providing an embedding of a countable free nilpotent group of any given class c into a finitely generated solvable linear group. The proofs of Corollary 2 and Corollary 3 follow immediately upon examining their linear representations provided in [8] and [9] respectively.

To begin note that every countable free nilpotent group of any given class c has a faithful representation in $\text{Tr}(n, k)$ for some positive integer n and some suitably chosen commutative field k . For example, let x be a single indeterminate over Z and for each $r = 1, 2, \dots$ let $x_r = (x_r(i, j))$ be an element of $\text{Tr}(c + 1, k)$ where k is the quotient field of the polynomial ring $Z[x]$ and

$$x_r(i, j) = \begin{cases} 0 & (i - j \geq 2 \text{ or } i - j \leq -1), \\ 1 & (i - j = 0), \\ x^{2cr+i-2} & (i - j = 1) \end{cases}$$

(here i denotes the row and j the column of the entry $x_r(i, j)$). Then $F = \text{gp}(x_r | r = 1, 2, \dots)$ is a free nilpotent group of class c on the generating set $X = \{x_r | r = 1, 2, \dots\}$ (see again B. A. F. Wehrfritz [17, p. 34]). It is easy to embed F in a finitely generated subgroup G of $\text{Tr}(c + 1, k)$. In fact let $A_{i+1, i}$ be the matrix in $\text{Tr}(c + 1, k)$ with entry 1 in the $(i + 1)$ th row and i th column, 1 everywhere on the main diagonal, zero row and i th column, 1 everywhere on the main diagonal, zero elsewhere ($1 \leq i \leq c$). For each j ($1 \leq j \leq c$) let D_j be the diagonal matrix in $\text{Tr}(c + 1, k)$ with entry x in the j th position on the main diagonal and 1 elsewhere. Let G be the subgroup of $\text{Tr}(c + 1, k)$ generated by the elements $A_{i+1, i}$ ($1 \leq i \leq c$), D_j ($1 \leq j \leq c$). An easy calculation shows that for each $r = 1, 2, \dots$

$$x_r = A_{2,1}^{D_1^{2^r}} A_{3,2}^{D_2^{2^{r+1}}} \cdots A_{c+1,c}^{D_c^{2^{r+c-1}}}.$$

Thus F is a subgroup of G . It follows that we have proved Corollary 1.

10. Finally we remark that V. N. Remeslennikov's example [14] of a finitely presented solvable group G without the maximal condition for normal subgroups is incorrect. In view of the importance of the problem involved we shall indicate how Remeslennikov's example fails. To do this we first recall that

$$G = \langle a, x, y; [x, y], a^y = aa^x, [a, a^x]^x = [a, a^x], [[a, a^x], a] \rangle.$$

It turns out that $G' = \text{gp}_G(a)$, the normal closure of a in G , is nilpotent of class 2 and that G is a split extension of G' by $B = \text{gp}(x, y)$ a free abelian group on x and y .

We let $K = \text{gp}_G([a, a^x])$ be the normal subgroup of G generated by the element $[a, a^x]$. The factor group G/K can be presented by $(\bar{w} = wK, w \text{ in } G)$,

$$G/K = \langle \bar{a}, \bar{x}, \bar{y}; [\bar{x}, \bar{y}], \bar{a}^{\bar{y}} = \bar{a}\bar{a}^{\bar{x}}, [\bar{a}, \bar{a}^{\bar{x}}] \rangle.$$

It follows from Lemma 2 that G/K is metabelian. Since G/K is finitely generated it satisfies the maximal condition for normal subgroups (Hall [11]). Therefore G satisfies the maximal condition for normal subgroups if and only if K satisfies the maximal condition for its G -invariant subgroups (a subgroup of K is G -invariant if it is a normal subgroup of G).

Let $L = \text{gp}([a^{x^i y^j}, a^{x^{i+1} y^j}] | i, j \text{ integers})$. Then L is a subgroup of the center of G' . Since every element of G has the form $x^m y^n g'$ where m, n are integers and g' is an element of G' , it follows that L is a normal subgroup of G . Because L contains the element $[a, a^x]$, $K = L$. We let $R = ZB$ be the integral group ring of B . Then R is noetherian. K becomes a right R -module in the usual way (the action is conjugation by elements of B extended to R in the natural way) and every normal subgroup of G contained in K is a right R -submodule of the right R -module K . But clearly K is finitely generated as a right R -module, indeed by $[a, a^x]$. Since R is noetherian, K is also noetherian. Hence K satisfies the maximal condition for its G -invariant subgroups and therefore G satisfies the maximal condition for normal subgroups.

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